THE
MATHEMATICAL GAZETTE.

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LOCAL BRANCHES.

THE LONDON BRANCH.

A MEETING of this Branch was held on Saturday, Nov. 2nd, at the London Day Training College. After a few introductory remarks from the Chairman, Mr. Abbott, a paper was read by Mr. Jackson on "The International Commission on the Teaching of Mathematics." Mr. Jackson endeavoured in this short paper to give a general impression of the problems raised in the 9000 printed pages of reports presented to the International Congress of Mathematicians at Cambridge last August.

Mr. Wallis, B.Sc., of the Holloway County School, read a paper on the "Correlation of the Teaching of Geography and Mathematics." Mr. Wallis urged that the properties of similar figures, the use of the sine, cosine and tangent, and the methods of contracted calculation should be taught by the mathematical master much earlier than is customary, as these portions of Mathematics are all required very early in the Geography course. An interesting discussion followed.

NORTH WALES BRANCH.

A MEETING of this Branch was held on November 9, at the Friars' School, Bangor, where Mr. Glynn Williams entertained the members present. Mr. E. H. Harper (University College) read a paper entitled "The Development of Geometry since Euclid, and some Ideas on its Teaching." After a brief reference to geometricians of early times, the paper proceeded to explain the non-Euclidean investigations and theories of such men as Gauss, Lobachewsky and Riemann, in the nineteenth century. In teaching geometry, Mr. Harper recommended concrete examples to begin with, and deductive and intuitive methods to be employed alternately. Prof. G. B. Mathews thought that no logical principle could be forced on beginners, and that intuitive methods were the best. He referred to the idea that Euclid's postulate about parallels is a property of matter, not of space. Mr. R. W. Jones regretted the absence of one definite text-book for younger children, but others considered that was not of much importance if the teacher was capable. It was pointed out that Germany had produced more distinguished mathematicians than England, although the former country had abandoned Euclid as a text-book for many years. Several members thought it would be beneficial if the Mathematical Association were to sanction a fixed sequence of propositions, or if one mathematician were selected to write an authoritative text-book.

THE INTERNATIONAL COMMISSION ON THE TEACHING OF MATHEMATICS.

ABSTRACT OF A PAPER READ BY MR. C. S. JACKSON, AT THE MEETING OF THE LONDON BRANCH.

1. As a first result of the labours of the International Commission on the Teaching of Mathematics, appointed in 1908 on the initiative of Professor D. E. Smith, the distinguished author of *Rara Arithmetica*, some 9000 printed pages were presented to the International Congress of Mathematicians at Cambridge. It is obviously impossible to go into details: I can only hope to mention one or two facts, and to give a general impression of the problems raised.

The abundant bibliographical and historical details worked out so carefully in the German reports enable a mental picture of the development of mathematical education during the last 100 years to be formed.

[A graph was shown from Dr. Gebhardt's report, illustrating the frequency of references in school text-books to the history of mathematics. The subject reached a minimum in 1850-60. Since then it has steadily risen, but has not yet reached the level of 1800-1810. Other graphs dealt with the number of schools using 4, 5, and 7-figure logarithms at different periods.]

The American reports are specially valuable for the care with which great masses of evidence had been collected and sifted. It is encouraging to find that our difficulties are not insular weaknesses, but are shared by our professional brethren in America, and indeed throughout the civilised world.

2. Leaving out the more technical questions, such as the place of the calculus, the necessity of facing the introduction of vector methods, and so on, a great question stands out which might be stated thus: What of the Perry methods?

It is the fate of every reformer to be caricatured by his followers. An eminent man with a broad view of his subject as a whole points out that a portion of it has been neglected. His followers see nothing but that portion. Professor Perry has always stood for sound mathematics, and the claim made by him for "Justice to the average boy" will never die. Professor Perry showed that the capacity of the average boy to deal with abstract reasoning had been over-estimated, and that the importance of concrete illustrations and of making good the claim of mathematics to rank with the humanities had been under-estimated.

But some of his followers are unjust to the boy of special ability. They scrapped mathematics as a coherent subject, and replaced it by shreds and patches. The signs of reaction are unmistakeable. The reaction will be led by technical institutions, roused by the deficient previous training of their students. We must see to it that the sound elements in reform are not swept away with the rubbish.

As to formal discipline, the American Commissioners asked, "If you regard mental discipline as an important aim, explain clearly exactly what you mean by mental discipline," and received one answer that should become classical, and be associated with the name of its framer.

"Mental discipline is that process of mind which

(*a*) recognises that there is a problem;

(*b*) wills that the problem be solved;

(*c*) perseveres until it is solved."

The American report conveys a weighty and impressive caution against hasty acceptance of loose dicta that the "doctrine of formal discipline is exploded," any precise statement of the "doctrine" being usually withheld: and against allowing the attempt to bring out the concrete side of mathematics, an attempt most right and needful, to degenerate into mere utilitari-

anism. But the impression which persists after the perusal of the reports is after all of the danger of giving advice. The advice has recently been given to teachers of geometry: "Do not pay too much attention to drawing." I admire the technical dexterity with which the precept is drawn up. Who can controvert it? But what must be the effect of such advice on a teacher conscious that he is no mathematician, and seeking for guidance?

It is said that Sir A. Clark once dismissed a patient—after directions as to weak tea, dry toast, and a return visit in a fortnight—with the impressive remark: "Now mind: *two* cigars a day." The patient reported on his return that he had followed all the directions, except that he could *not* get through those two cigars a day.

Is it not a fact that the best advice for *A* might be to pay a great deal more attention to neatness, and to *B* not to let the sense of neatness he had aroused expend itself in ruling red lines and margins, but to make it grow to a sense of geometric elegance? Unfortunately the advice you meant for *B* is so often taken by *A*. Enthusiastic and indiscreet followers of Professor Perry seize on advice meant for the man who teaches exactly as his own schoolmaster did forty years ago, and *vice versa*. Of course, this question of the value of general advice is not all on one side. Some years ago the Headmaster of Eton published an appeal to all who read Plato with their feet on the fender to rally to Cambridge for the defence of Little-Go Greek.

Here is a beautiful instance of a kind of reasoning common among mathematicians! "I have for five and twenty years been thinking about tangential coordinates, or the axioms of algebra, or non-Euclidean geometry, or . . . or . . . therefore boys of fifteen should study the subject, and will find in it all the solace and strength that I have found." It is fatally easy to deceive oneself as to the extent to which pupils appreciate refinements in mathematical analysis. And yet, after all, it is better to aim too high than too low. The sober prejudice of Englishmen, which De Morgan praised, has led to the postponement of valuable reforms, but it has also saved us from throwing away that which was, and is, good. The problem of combining a coherent and logical scheme of mathematical education with due attention to the average boy, and to the value of topics arising from familiar daily life in developing interest and an appreciation of accuracy, is a difficult one, but it is in process of solution. The labours of the Commission—in particular of the German and American contributors—have forwarded the process of solution in many details.

I am conscious that the task of saying anything useful in twenty minutes on so great a mass of material is beyond my powers. I may hope, however, to have called your attention to the existence of a storehouse of valuable matter.

THE TEACHING OF LIMITS AND CONVERGENCE TO SCHOLARSHIP CANDIDATES.

II.

1. It seems most convenient, in virtue of the well-established pedagogic maxim that "algebraic subjects should, as far as possible, be introduced as generalisations of known principles in Arithmetic," to base our teaching of limits and convergence upon the arithmetic theory of "Approximations," with which all scholarship candidates are familiar as a practical instrument.

The formal definition of a limit by means of a sequence is usually found by the beginner very hard to understand, but if presented from the point of view of approximations it is less difficult.

By way of illustration, let us take the case of the limit π. Suppose that a country joiner is fitting a lid into the mouth of a tar barrel. It will, in

general, be sufficient for his purpose to say that the circumference of the lid of the barrel is 3 times its diameter. A cabinet-maker, who does much finer work, would require a higher degree of accuracy in the exercise of his trade, and might take the circumference of a circle to be $\frac{22}{7}$ of the diameter. The jeweller would require even a still closer approximation. If, therefore, a systematic tabulation of approximate values for π were attempted, it would be convenient to give the first few approximate values and then to append the law of formation of these approximate values, so that any mechanic would have the means of computing an approximation to as great a degree of accuracy as he pleased. Thus, by way of illustration, the table might be set forth as follows :

GREGORY'S TABLE OF APPROXIMATION FOR π.

$$\pi_n = 4\left\{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\ldots+(-1)^{n-1}\frac{1}{2n-1}\right\}.$$

Label.	Members of Sequence.
1	$\pi_1 = 4$
2	$\pi_2 = \frac{8}{3}$
3	$\pi_3 = \frac{52}{15}$
4	$\pi_4 = \frac{304}{105}$
etc.	etc.

Various other tables for carrying out this approximation to any required degree of accuracy have been devised.

One other point has to be noticed. The mechanic who is using such tables as those described above has the right to expect that every subsequent member of the sequence gives *at least* as close an approximation as any one in front. Thus he has the right to expect that every member after, say, π_{15} is *at least* as good an approximation as π_{15} ; in other words, that π_{23} differs numerically from π by at least as little as π_{15} or π_{17}. Of course, *in general*, if the sequence $s_1, s_2, s_3, \ldots s_n, \ldots$ forms a tabulated sequence giving successive approximations to the limit s, s_{n+1} will be a closer approximation to s than s_n. Sometimes, however, s_{n+1} may be equal to s_n, but at all events the arithmetical difference between s_{n+1} and s must not exceed the arithmetical difference between s_n and s.

To give a practical illustration of a case when two successive members of an approximation are equal, we need only consider the approximation in decimals to $\sqrt{1{\cdot}0001}$. We have the tabulation :

Label.	Members of Sequence.
1	$s_1 = 1$
2	$s_2 = 1{\cdot}0$
3	$s_3 = 1{\cdot}00$
4	$s_4 = 1{\cdot}000$
5	$s_5 = 1{\cdot}0000$
6	$s_6 = 1{\cdot}00004$
7	$s_7 = 1{\cdot}000049$
etc.	etc.

Here $s_1 = s_2 = s_3 = s_4 = s_5$.

We are now in a position to state the formal definition of how a sequence is to define a limit:

The sequence $s_1, s_2, s_3, s_4, \ldots s_n, \ldots$ *(where the law of formation is explicitly stated) is said to define the limit* s *(or to form a perfect approximation to the number* s*) if, however small be the positive number* ϵ *arbitrarily assigned, we can determine a positive integer* n_ϵ *corresponding to* ϵ *such that* $s_{n_\epsilon}, s_{n_\epsilon+1}, s_{n_\epsilon+2}$, *etc., each differ from* s *by less than* ϵ, *however far along the sequence beyond* s_{n_ϵ} *we care to go.*

Thus the country carpenter in his approximation to π would use only a moderately small value for ϵ, and he would not have to go very far along the sequence $\pi_1, \pi_2, \pi_3, \ldots \pi_n, \ldots$ until he came to a member π_{n_ϵ} which (as well as all subsequent members) differed from π by less than ϵ, or, colloquially, until π_{n_ϵ} (and, of course, all subsequent members) gave a sufficiently close approximation for his purpose. The ϵ of the cabinet-maker would, in general, be smaller than that of the country joiner, and the ϵ of the jeweller smaller still.

2. Hitherto we have been supposing that we were given definite numbers such as π, $\log 2$, $\sqrt{2}$, etc., and were finding sequences approximating to them to any required degree of accuracy. The converse problem is of even greater importance:

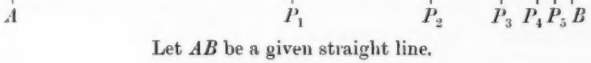
Given a sequence $s_1, s_2, s_3, \ldots s_n, \ldots$ *obeying a definite law, does it form a perfect approximation to any definite number? In other words, under what circumstances does a given sequence define a limit?*

There are many instances in which it is perfectly plain that the given sequence does not form a perfect method of approximation to a definite number, *i.e.* does not define a limit. Thus, consider

$$\begin{aligned} s_1 &= 1, && (=1)\\ s_2 &= 1-1, && (=0)\\ s_3 &= 1-1+1, && (=1)\\ s_4 &= 1-1+1-1, && (=0)\\ &\ldots\ldots\ldots\ldots\\ s_n &= 1-1+1-1+\ldots+(-1)^{n-1}1,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

There is no steady approximation to a definite number in this case. So it is with $s_n = \sin n\frac{\pi}{2}$.

Again, consider the following successive operations, where plainly a limit is defined.



Let AB be a given straight line.
Bisect AB in P_1.
Bisect P_1B in P_2.
Bisect P_2B in P_3.
.....
Bisect P_nB in P_{n+1}, and so on.

We thus get a sequence of points $P_1, P_2, P_3, \ldots P_n, \ldots$. They obviously form a perfect approximation to the definite point B, or, in more academic language, the sequence of points $P_1, P_2, P_3, \ldots P_n, \ldots$ defines the limit B. This result is intuitive, but we can by way of verification apply the formal test given in the definition of Art. 1.

Let $AB=2$ cms.

$$\therefore\ P_1B=1 \text{ cm.},$$

$$P_2B=\frac{1}{2} \text{ cm.},$$

$$P_3B=\frac{1}{2^2} \text{ cm.},$$

$$P_4B=\frac{1}{2^3} \text{ cm.},$$

$$\ldots\ldots\ldots\ldots\ldots,$$

$$P_nB=\frac{1}{2^{n-1}} \text{ cm.},$$

$$\ldots\ldots\ldots\ldots\ldots$$

Now $P_nB \geqq \epsilon$ (where ϵ is any definite positive number, however small, given beforehand)

$$\text{if } \frac{1}{2^{n-1}} \leqq \epsilon; \quad i.e. \text{ if } 2^{n-1} \geqq \frac{1}{\epsilon},$$

and it is always possible to so choose n_ϵ that

$$2^{n_\epsilon-1},\ 2^{n_\epsilon},\ 2^{n_\epsilon+1},\ 2^{n_\epsilon+2}, \ldots \text{ and all subsequent numbers are } \geqq \frac{1}{\epsilon}.$$

E.g. if ϵ were given as $\frac{1}{10000}$,

$$2^{n-1} \text{ would have to be } \geqq 10000.$$

But $10^4=10000$;

$\therefore$ clearly $16^4>10000$,

i.e. $2^{16}>10000$.

Hence it is plain that for $\epsilon=\frac{1}{10000}$,

the previously defined n_ϵ can be taken as $=17$,

since $P_{17}B,\ P_{18}B,\ P_{19}B \ldots$ are each $<\frac{1}{10000}$.

N.B.—It does not necessarily mean that the 17th is the *first* member of the sequence for which $P_nB<\frac{1}{10000}$. In theory it is sufficient to shew that such a member $P_{n_\epsilon}B$ can be found; in practice it is expedient to find the *first* member of the sequence satisfying the given conditions in order to save the labour of an unnecessary degree of approximation.

Again, if we started with a tumblerful of wine, poured out half the wine and added water, again poured out half the contents of the glass and filled up with water, and so on as often as we pleased, we are plainly defining the limit zero as far as the wine in the tumbler is concerned; for if 1 designate the amount of wine in the tumbler to begin with, we shall as we proceed have amounts of wine designated by $1, \frac{1}{2}, \frac{1}{2^2}, \ldots \frac{1}{2^n}, \ldots$.

The following is a beautiful geometrical illustration due to Mr. F. J. W. Whipple.

To approximate to the trisection of an angle using only Euclidean methods.

Let AOB be the given angle.

Bisect AOB by OP_1.
" AOP_1 " OP_2.
" P_1OP_2 " OP_3.
" P_2OP_3 " OP_4.
" P_3OP_4 " OP_5.
" etc. etc.

The sequence of angles AOP_1, AOP_2, AOP_3, ... AOP_n, ... form a perfect approximation to the angle $AOP=\frac{1}{3}AOB$.

To prove this, we see that

$$AOP_n=AOP_1+P_1OP_2+P_2OP_3+P_3OP_4+\ldots+P_{n-1}OP_n$$

(each taken with its proper sign)

$$=AOB\left\{\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}+\ldots+(-1)^{n-1}\frac{1}{2^n}\right\}=\frac{AOB}{3}\left\{1-(-1)^n\frac{1}{2^n}\right\}.$$

Hence *the numerical difference* between AOP_n and $\frac{1}{3}AOB$ is $\frac{AOB}{3.2^n}$, and if we apply the formal test of Art. 1, we see that this is $<\epsilon$ if $2^n>\frac{AOB}{3\epsilon}$, and this affords a means of finding n_ϵ that is plainly possible.

The formal test is therefore satisfied, and we have thus another example of a limit being uniquely defined by means of a sequence.

3. It must now be stated that, in very many cases, numbers must of necessity be defined by means of sequences, and we have to find out all about the properties of these numbers not by manipulating the numbers themselves, but by manipulating the sequences defining them. Thus we can investigate the properties of such numbers as 2, 3, $\frac{1}{3}$, ·056 by applying to them direct methods, but the application of such direct methods becomes inconvenient or even impossible in the case of such numbers as π, e, $\log_e 7$, etc.

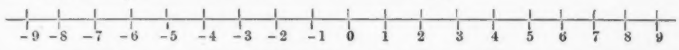
We may compare the methods of investigation to be adopted in the two cases to the methods of finding the mean density of a brick and the mean density of the earth. We apply the direct method of the balance to the brick, but we have to resort to indirect measures and tentative means in the case of the earth. So it is with numbers. Some numbers can be directly handled, some can not; these latter have to be dealt with through the instrumentality of sequences. One would never dream of investigating the properties of $\frac{1}{3}$ by an appeal to the sequence s_1, s_2, ... s_n, ... where

$$s_n=\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\ldots+(-1)^{n-1}\frac{1}{2^{n-1}},$$

but in the case of π, e, γ, etc., such an indirect *modus operandi* is inevitable.

We must therefore investigate some of the general properties of limits and their defining sequences, and we shall adopt Dedekind's method.

4. We begin by making quite clear to the pupil what a rational number is. It is the ratio of two integers, positive or negative, *e.g.* $\frac{3}{4}$, $\frac{-5}{6}$, $-\frac{7}{8}$, ·459, etc. We represent these in a line $x'0x$ in the usual way.



In order to represent graphically the number infinity, it is convenient to use a circle rather than a straight line.

The diagram makes this quite clear. It has the advantage of representing graphically the number ∞ as a definite point and not as an unexplored region of the line $x'0x$ which is too often the vague interpretation put on ∞ by the beginner.

The state of affairs presumed on the part of the student now is this:

(1) He knows about the properties of rational numbers.

(2) He represents rational numbers by points on the above line (or circle), as explained above.

(3) All rational numbers are represented by points on the line (or circle).
(4) Do all points on the above line (or circle) represent graphically rational numbers?

Question (4) is answered by Pythagoras's Theorem very simply. Draw a triangle ABC, right-angled at B and having $BC=BA=1$ unit of length. Mark off on $x'0x$ a length $0l$ equal to AC. Does l represent a rational number? If l represents a rational number, then AC must contain $\frac{p}{q}$ units of length, where p and q are positive integers. Suppose that fraction $\frac{p}{q}$ is in its lowest terms. Then, by Pythagoras's Theorem,

$$\frac{p^2}{q^2}=2\,;$$

$$i.e.\ p^2=2q^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now, if p be an odd integer, (1) is plainly impossible.
Hence p must be even.
Let

$$p=2r\,;$$

$$\therefore\ q^2=2r^2.$$

Plainly, as before, q cannot be odd, *i.e.* q is even. But if p and q are both even integers, $\frac{p}{q}$ is not a fraction in its lowest terms. Hence the hypothesis that l represents a rational number breaks down. Many other points that do not represent rational numbers on the line (or circle) can easily be obtained in a similar way. We therefore divide the points on the line (or circle) into two classes:

I. Points that represent graphically rational numbers. We call these points Rational Points.

II. Points that do not represent graphically rational numbers. We call these points Irrational Points.

Now, if we revert to the case of the hypotenuse of a right-angled isosceles triangle, we saw that, when we attempt to measure it, we cannot give its length in terms of rational numbers, the side of the triangle being the unit, but we can give the length as approximately as we please by means of sequences in terms of rational numbers. We therefore extend the definition of "number," and say that such lengths are measured by "irrational numbers."

5. Consider now any irrational point l on the line $x'0x$. As l is not itself a rational point, it plainly divides all the rational points into two classes, (1) the rational points lying to the left of l, (2) the rational points lying to the right of l. Also every point of Class (1) lies to the left of every point of Class (2).

If, on the other hand, l is a rational point on the line $x'0x$, it plainly divides all the rational points into two classes, (1) the rational points lying to the left of or coincident with l, (2) the rational points lying to the right of l. Plainly, every point of Class (1) lies to the left of every point of Class (2).

Dedekind's theory of irrational numbers may be said in brief to consist of "*The properties of rational numbers* **plus** *the assumption of the converse of the above proposition based on intuition.*"

Dedekind's Hypothesis (*or Intuition*).

If p and q be two rational points on the line $x'0x$ and if all the rational points in the segment pq be divided by some assigned law into two classes such that every member of Class (1) *lies to the left of every member of Class* (2), *then there exists one and only one point l acting as the boundary between the members of the two classes.*

Note I. This constitutes the definition of the point l, called by Dedekind the *schnitt* or *cut*.

Note II. l may be either a rational or an irrational point, and this can be decided only by a knowledge of the law defining the members of the two classes.

Example 1. Consider all the rational points lying in the segment $0, \infty$. Let us divide them into two classes according as the squares of the rational numbers they represent are $\leqq 2$ for Class (1) or >2 for Class (2). It is plain that every member of Class (1) lies to the left of every member of Class (2). Hence, by Dedekind's Hypothesis, a definite point exists separating the two classes.

The student will see that this constitutes the definition of the point representing $\sqrt{2}$. The schnitt is in this case an irrational point.

Example 2. Consider all the rational points lying in the segment $0, \infty$

The behaviour of the sequence $\left(\frac{3+5n}{4+7n}\right)$ divides them into two classes, as is evident from the following considerations:

The first few members of the sequence are $\frac{3}{4}, \frac{8}{11}, \frac{13}{18}, \frac{18}{25} \ldots$. The sequence is plainly a steadily decreasing one, since

$$\frac{3+5n}{4+7n} > \frac{3+5(n+1)}{4+7(n+1)}.$$

Also, if n can be found so large that *one* member of the sequence lies to the left of a given rational point, all the subsequent members will lie to the left of that point. Now, if we consider all the rational points lying in the segment $0, \frac{3}{4}$, we can find some such that the above sequence passes to the left of them if n be chosen big enough and others which the above sequence never passes to the left of, however big we take n.

Thus
$$\frac{3+5n}{4+7n} < \frac{2}{3}$$

if
$$3(3+5n) < 2(4+7n),$$

i.e. if
$$n < -1.$$

This is never the case, and hence the members of the given sequence never attain to the left of $\frac{2}{3}$.

We can therefore divide the rational points (or numbers) in the interval $0, \frac{3}{4}$ into two classes:

Class (1). The rational points which the above sequence never passes to the left of, however big n may be chosen.

Class (2). The rational points which the above sequence does pass to the left of, if n be chosen large enough.

Plainly every member of Class (1) lies to the left of every member of Class (2).

There must, therefore, be a schnitt or cut-point separating the two classes by Dedekind's Hypothesis. The schnitt is evidently the limit which the sequence $\left(\frac{3+5n}{4+7n}\right)$ decreases down towards but never actually reaches, viz. $\frac{5}{7}$, as may be found by the direct test of Art. 1.

6. *There are rational points as near as we please to any given irrational point.*

Let l be an irrational point.

Suppose we wish to prove that there are rational points which are so situated as to differ from l by less than ϵ, where ϵ is rational.

Imagine an unlimited number of matches at our disposal, each cut into length ϵ. Place these end to end. Plainly, if we proceed long enough, we can place sufficient matches along the line $x'0x$, the first one having its left end at 0, that we shall pass any marked point on the line. If, therefore, l lies between $n\epsilon$ and $(n+1)\epsilon$, either of the rational numbers $n\epsilon$ or $(n+1)\epsilon$ differs from the irrational number l by less than the rational number ϵ.

7. *Any irrational number can be approximated to by a decimal approximation.*
Let l be the irrational number.

Take matches of unit length. We can then find by trial the two integers between which l lies. Next take matches 10^{-1} of a unit in length. We can then find by trial the two consecutive decimals of the first order between which l lies. Next take matches 10^{-2} of a unit in length. We can then find by trial the two consecutive decimals of the second order between which l lies.

We can proceed in this way to any required degree of approximation.

(*To be continued.*)

A CASE OF THREE ROTATING LINES AND THE POINT "O."

I.

1. Let AP, BQ, CR be three lines through the vertices of the $\triangle ABC$ which rotate with equal angular velocity ω, and which make equal angles with the opposite sides, the initial position being that of the three altitudes of ABC. Then the three circles AQR, BRP, CPQ, which meet in the well-

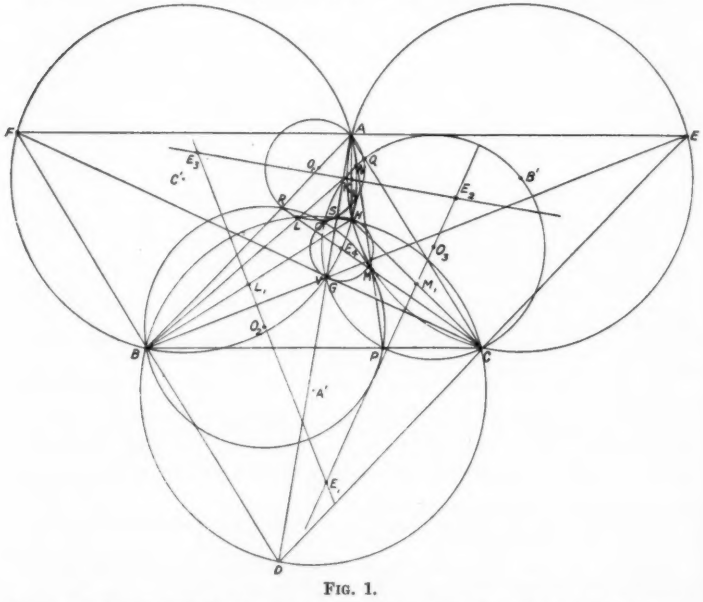


FIG. 1.

known point O, will be shewn to form three coaxal systems; and among other things we shall shew that the locus of O is the circle on GH as diameter, G being the centroid and H the orthocentre.

2. Let (BQ, CR), (CR, AP), (AP, BQ) be the points L, M, N. These points lie on the three circles respectively, and if DEF be the $\triangle$ formed by parallels

through A, B, C to the opposite sides, the points L, M, N lie on the three equal circles BCD, CAE, ABF, these latter being concurrent at H.

It therefore follows that since (by hyp.) the angle $HAM=HBN=HCL=\theta$ say, we have $HL=HM=HN$, or H is the circumcentre of LMN. Also the $\triangle LMN$ is evidently similar to ABC and DEF. Moreover H is the circumcentre of DEF, which is itself one of the members of the family LMN.

The circles LMN therefore form a concentric system which expands and contracts, the radius varying from nothing to HD while the three lines AP, etc., have turned through a right angle.

We may note, too, that the points L, M, N move with angular velocity 2ω about the centres A', B', C' respectively, or with angular velocity ω about H collectively.

3. Let AD meet the circle A' in S, and let T, V be corresponding points for BE and CF.

Then $\angle SLQ=SDB=SAQ$. Hence the variable circle $ARLQ$ passes through the fixed points A and S, thus forming a coaxal system. Similarly for the other two circles.

The three radical axes are the medians of ABC, but obviously G is not a radical centre.

Since HD, HE, HF are diameters, $\therefore HS \perp SG$, etc.; and hence S, T, V lie on the circle, diameter GH.

It is seen that $\triangle STV$ is similar to the $\triangle$ formed by the medians.

4. To shew that O lies on this circle GH, we have (see fig. 2)

$$\angle SOR=\pi-GAB,\ TOR=\pi-GBA;$$

$$\therefore\ \angle SOT=GAB+GBA=SGT,$$ which proves the statement.

II.

5. Let d_1, d_2, d_3 be the diameters of the circles AQR, etc., say (O_1, O_2, O_3). Then, since $\angle ARO=\pi-ASO=OSG=OHG$,

$$\therefore\ AO=d_1 \sin OHG=\frac{d_1 . OG}{GH};$$

$$\therefore\ d_1=\frac{AO}{OG} . GH;\ \text{whence also } d_1 : d_2 : d_3=AO : BO : CO.$$

6. The sides of the $\triangle O_1O_2O_3$ are $\perp$ to OP, OQ, OR, the common chords. Hence $\angle O_1=\pi-ROQ=A$; or the $\triangle O_1O_2O_3$ is similar to ABC. The $\triangle O_1O_2O_3$ is therefore one of a system of $\triangle^{les}$ of given species (ABC) inscribed in the $\triangle E_1E_2E_3$, formed by the lines of centres, which latter is similar to the $\triangle$ formed by the medians of ABC.

7. Since $O_1O=O_1A$ we have

$$AO_1 : BO_2 : CO_3=OO_1 : OO_2 : OO_3=d_1 : d_2 : d_3=AO : BO : CO;$$

$$\therefore\ \triangle^{les} AOO_1,\ BOO_2,\ COO_3 \text{ are similar.}$$

Hence the point O is the centre of similitude (c.s.) of $\triangle^{les}$ $O_1O_2O_3$ and ABC.

8. Again, by Art. 6 and the "point O theorem," the $\odot^{les}$ $E_1O_2O_3$, $E_2O_3O_1$, $E_3O_1O_2$ meet in a *fixed* point E_4, which is the c.s. of all the $\triangle^{les}$ of the system, $O_1O_2O_3$.

9. To find E_4, we have $\angle O_2E_4O_3=\pi-E_1=BGC$, etc. But $\triangle O_1O_2O_3$ is similar to ABC; hence E_4 corresponds to G, and is therefore the common centroid of all the $\triangle^{les}$ $O_1O_2O_3$.

Again, the $\triangle K_1L_1M_1$ formed by the mid-points of HA, HB, HC is similar and similarly situated to ABC; it is also the initial position of $O_1O_2O_3$. E_4 is therefore the centroid of $K_1L_1M_1$, and hence corresponds to G of $\triangle ABC$. But H is the c.s. of $K_1L_1M_1$ and ABC. $\therefore E_4$ is mid-point of HG; that is the $\odot^{re}$ of the circle GH.

10. We may note that since $E_4K_1 \parallel GA \perp E_2E_3$, and similarly for E_4L_1, E_4M_1; $\therefore K_1L_1M_1$ is the minimum $\triangle^{le}$ of the system $O_1O_2O_3$.

III.

11. Since E_4O_1 is the $\perp$ bisector of SO, it follows that E_4O_1 rotates with the same angular velocity as SO, *i.e.* with the angular velocity of O about any fixed point (say G) on the circle GH.

In other words, the $\triangle O_1O_2O_3$ rotates about E_4 with the angular velocity of GO about G.

12. Now (figs. 1, 2) $\angle NOC = NPC = \pi - BPM = BOM$;

$$\therefore \angle MON = BOC.$$

Similarly $\angle NOL = COA$, $\angle LOM = AOB$;

$\therefore$ O is the c.s. of $\triangle^{les}$ LMN, ABC.

But O is the c.s. of $\triangle^{les}$ $O_1O_2O_3$ and ABC;

$\therefore$ O is the c.s. of $\triangle^{les}$ $O_1O_2O_3$, LMN, and ABC.

13. Now in $\triangle^{les}$ ABC, LMN, since the circumcentres are corresponding points,

$\therefore$ O' corresponds to H; and, O being the c.s.,

$$\therefore \frac{OO'}{OH} = \frac{BC}{MN},$$

and $\angle O'OH = \angle^{le}$ between BC and $MN = APC$;

$$\therefore \angle O'OG = O'OH - \frac{\pi}{2} = APC - \frac{\pi}{2} = HAP \equiv \theta.$$

14. Let $\angle OGH = \phi$.

Then
$$\frac{OH}{O'H} = \frac{\sin OO'H}{\sin O'OH} = \frac{\sin \overline{\phi - \theta}}{\cos \theta},$$

or, if K = radius of circle GH,

$$2K \sin \phi \cos \theta = 3K \sin \overline{\phi - \theta},$$

whence $\tan \phi = 3 \tan \theta$, giving a relation between the angular velocity of GO and that of AP, viz. ω or $\dot{\theta}$. Hence the motion of O round its circle is determined.

(To be continued.)

REFORM OF MATHEMATICAL TEACHING IN GERMANY.*

I PROPOSE in this paper to deal with the development during the last twenty or thirty years of the dominating ideas of the reform of mathematical teaching in Germany, and must beg for your indulgence if the theme is one which is already familiar.

In the *Mathematical Gazette* of December last there appeared a translation of the so-called Meraner Lehrplan of 1905, which, with the appended notes, formed a kind of Declaration of Rights of the Reform Party. It is my intention to give some account of the movement which led to its publication, to indicate briefly the substance of these appended notes, and to give some details of the way in which the general ideas have been introduced into the curricula.

The most remarkable feature in this development is the manner in which the contentions of the first reformers have been so largely accepted and pressed forward not only by mathematicians but by men of general culture whose bent has been towards humanistic or scientific rather than purely

* Paper read before the London Branch of the Mathematical Association by E. Allan Price, B.A.

mathematical studies—a fact which speaks for the catholic nature of German education in the past.

To one who is acquainted with Young's admirable little book on Prussian Schools much of my preliminary remarks must appear redundant, but in case some of those present are not conversant with the organization and methods in vogue I venture to trouble you with some facts and figures.

In order to understand clearly the movement for reform in the mathematical teaching in Germany, it is essential to have a clear conception of the various classes of schools, their aim and courses. The names employed convey such a much more definite idea of the essential characteristics than do our English terms, that I propose, at the risk of trying the patience of my hearers, to give a short account of these different types of school.

With Elementary Schools as we understand the terms I am not concerned, but shall deal with those institutions which in England we sometimes describe by the name of Secondary Schools, including in this term Public Schools, Grammar Schools, and the modern Secondary Schools maintained by the local authorities; that is to say, those which provide a liberal education for the directing classes among the population.

In Germany as in England there grew up a feeling of dissatisfaction with the classical education provided by the older institutions, a feeling which resulted in the foundation of the modern schools (Realschulen), whose curricula dispense with Greek and Latin, substituting modern for ancient languages, and at the same time giving more attention to Mathematics and Science, though languages are recognized as the basis of the course in every school.

It is difficult for us to realize the extent to which schools are controlled by Government authority in Germany, but there is at the same time this advantage for the foreigner: that the definite regulations with the accompanying nomenclature enable us to obtain a clearer view of the system as a whole than is possible with the chaotic independence of schools in England.

The aim of Secondary Education is of course preparation for the University, but there is another factor which determines the type of secondary school demanded by public opinion in Germany, and that is the much-coveted privilege of serving one year in the Army instead of two, a privilege which can only be obtained by attaining the requisite standard in the proper type of school. Thus it comes about that in addition to the schools which cater for the Universities, there are others of the same type whose highest classes correspond with the 4th class of the 9-class schools preparing for the University. Boys sometimes pass from these schools to those which provide a more extended course, but the majority of the pupils look forward to the certificate which confers the privilege of one year's military service.

	9-Class Schools.	6-Class Schools.	Approx. Age.
Lower Stage.	Sexta, - - - VI	Sexta, - - 6	8-9
	Quinta, - - - V	Quinta, - - 5	9-10
	Quarta, - - - IV	Quarta, - - 4	10-11
Middle Stage.	Unter-Tertia, - - U III	Tertia, - - 3	11-12
	Ober-Tertia, - - O III	Secunda, - 2	12-13
	Unter-Secunda, - U II	Prima, - - 1	13-14
Upper Stage.	Ober-Secunda, - O II		14-15
	Unter-Prima, - - U I		15-16
	Ober-Prima, - - O I		16-17

Of the complete schools with 9-classes there are three types: the Gymnasium, the Realgymnasium and the Oberrealschule. Historically and socially the most prominent of all schools in Germany is the *Gymnasium*, the counterpart of our classical school, formerly the only possible road to the University, and still to-day the most usual one. The curriculum of the Gymnasium is based on the two classical languages with one obligatory modern language; that of the Oberrealschule on the modern languages, the intermediate Realgymnasium giving its main lines of instruction in Latin and two modern languages. Corresponding to these types of 9-class schools there are the three 6-class schools, Progymnasium, Realprogymnasium, and Realschule, the course being roughly the same as in the former, but omitting the three top classes.

In speaking of Germany one is apt to forget the independence of the various states of which she is composed, and the consequent variety in the organization of school. Prussia is the dominant partner in the confederacy, and most of the other states have followed her lead in the matter of education, though Saxony has shewn considerably more enterprise in some directions. But it is with Prussia and the courses prescribed for her schools that we have chiefly to deal.

There is one consideration which must be borne in mind in endeavouring to obtain a clear notion of the work of these schools as compared with that of corresponding institutions in England. I mean the special training of the teachers, who are not only all experts in Mathematics, but have undergone a prolonged training under the eye of men of experience of teaching in that particular branch. The consequence is that every mathematical master is accustomed to discuss his subject from every point of view, and each man feels confident that his knowledge of matter and method is sufficiently wide to make him a competent critic.

The essence of the spirit of Reform in Mathematical Teaching which is at present agitating all civilized countries is the attempt to introduce at an early stage the idea of variable quantities and functions, and the permeation of the whole mathematical course by these notions treated in a commonsense manner with concrete quantities. That is to say, that x and y should be regarded from their first introduction not only as unknown quantities to be determined, but also as variable quantities whose course, as their values continually alter, should be studied as a whole.

It may be said that what is contemplated is the introduction of the fundamental principles of Analytical Geometry at the beginning of the school course instead of being treated as a special subject at the end. But it must be remembered that in Analytical Geometry as taught formerly an equation between x and y is looked upon merely as a locus for points, and the particular values of the variables x and y are the chief concern; whereas the proposed reform insists upon the notion that in consequence of a change in the value of x there is a certain definite and consequent change in the value of the dependent variable y.

The contention of the reformers is that the idea of function defined in this way should be accepted as the central notion of *all* mathematical teaching, and an attempt be made to get a clear insight into the fundamental idea of the variable by means of simple concrete instances represented graphically. Now, this idea is by no means new, though until lately it has not been treated as a question of practical politics. My object in this paper is to give some account of the way in which, from continual discussions as to the admission of Analytical Geometry into the school course and its proper place in the curriculum, this radical alteration of the whole aim of mathematical teaching has been evolved.

In the official Programme (Lehrplan) for Prussia of 1816 the ideal ultimate standard of the school course was a very ambitious one, including Analytical Geometry, Maxima and the Calculus. But as the standard for matriculation

at the University was at the same time very much lower, this proved to be a dead letter. In 1834, however, Conics and Spherical Trigonometry were expressly forbidden except in the case of small special classes for particularly able mathematicians. There appeared in 1843 a text-book on Conics and a collection of Problems for schools, the work of a Berlin schoolmaster, *Schellbach*, who nevertheless had no intention of introducing the fundamental function-idea in any but the top class; only in the treatment of Maxima did he make use of the idea of the variable, avoiding all reference to graphical representation. At the same time he circumvented the procedure of the Calculus by the method since known in Germany by his name, perhaps to save himself from the accusation of overstepping the prescribed limits.

Another schoolmaster, *Balzer*, at Dresden, taught Analytical Geometry and Calculus in the year 1865 to a special class, and introduced the notion of function into his *Elements of Mathematics*, but without any attempt at graphical representation. He merely says in a note, "The line upon which the extremities of the ordinates lie is characteristic of the function," and does not even use a graph in connection with the Ratios in Trigonometry.

In the year 1864 there was a notable meeting of teachers in Hanover, where the whole question of the range of school mathematics came to the fore owing to a demand on the part of the humanists for a diminution in the time devoted to the subject. At this conference it was proposed to limit school mathematics to the consideration of constant quantities—statical as opposed to dynamical or variable quantities. *Bertram*, a teacher in a modern school in Berlin, pleaded for Analytical Geometry, but found no support; in his school, said his opponents, where Mathematics was taught eight or nine hours a week and the Calculus treated in the upper classes, it might be possible, but was quite out of the question in the Gymnasium.

At this time there was a remarkable revival in all departments of life owing to the successful wars of 1866 and 1871 and the great territorial acquisitions of Prussia, and one of the problems which engaged public attention was the position of the modern Real school, which claimed equal rights and privileges with the old-established Gymnasium. In the endeavour to give these modern schools a greater bias towards scientific studies, demands were made that, in addition to the Analytical Geometry already admitted for the top class in 1859, the Calculus should be introduced. At the same time it was widely felt that the true goal of the mathematical teaching in the Gymnasium also had not really been determined, and this found expression in the school Conference of 1873 summoned by the Prussian Minister of Education to discuss various questions of interest. It was in this conference that *Gallenkamp*, headmaster of a large Modern School in Berlin, had the temerity to say "that the task set before the Gymnasium in the matter of general culture demands the introduction of Analytical Geometry and Calculus into the curriculum; only by this means can the freshman at the University obtain an insight into the great work of Culture in the regions of Natural Science, and only thus can the serious and ever-widening breach between the two classes of educated men of the nation be avoided." His proposals were looked upon as revolutionary, and were opposed not only because of their alleged impracticability, but also on the ground that the scope of school Mathematics ought to be determined, not by the circumstances of the few privileged 9-class schools, but by those of the 6- or 7-class schools, which were at that time much more numerous.

A few years later (1877) another voice was raised on the same side, but for widely different reasons, producing an even greater flutter in the dovecots. Professor *du Bois-Reymond*, the celebrated physiologist at the University of Berlin, made a vigorous attack upon the Prussian Lehrplan, especially in Mathematics. As an enthusiast for classical training in the sense of cultivating the hellenistic spirit, he deplored the fact that the Gymnasium

no longer gave the best preparation for studies in *all* departments, but bade fair to surrender the field to the modern Realschule in some of them. He demanded an improvement in the teaching of Mathematics. "Analytical Geometry," says he, "is still excluded from the Gymnasium, though the curriculum of the Realschule goes far beyond it. This I believe to be a serious mistake. The study of Mathematics first really evolves its true cultural capacity in the transition from elementary subjects to analytical geometry. The representation of functions by curves or surfaces opens a new world of ideas, and teaches the use of one of the most productive methods by which the human mind has increased its own power." Sensational as was this pronouncement on the part of one whose eminence and disinterestedness commanded universal respect, there were but few who actively favoured the introduction of Analytical Conics into the school course; among these few, however, was *Hoffmann*, who, in his capacity as editor of the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* for the teaching of mathematics and natural science, was for many years a doughty champion in the cause of these subjects in the school curriculum. Not only did he demand this as an essential reform in the work of the Gymnasium, but he insisted on a minimum of four hours a week for Mathematics throughout the course. He was, however, not prepared to follow Gallenkamp in his demand for the Calculus in the Gymnasium, a subject which for some time had been admitted into certain Real-Schools.

It may then be said that in 1880 public opinion on the whole was opposed to the introduction of Analytical Conics in the secondary schools, though the treatment of Conics synthetically was supported in many quarters. At the same time it should not be forgotten that the question at issue was the introduction of Analytical Geometry in the highest classes, and that the ideal of the modern reformers had not yet been formulated.

At last the Prussian Lehrplan of 1882 made its long-awaited appearance, no essential alterations being made in the curriculum then in force. But though no systematic treatment of Analytical Geometry was allowed, yet under suitable conditions the elements of Conics might be treated analytically, "for by this means an idea of the Differential Coefficient may be obtained." At the same time in the Oberrealschule (the 9-class modern school) both Analytical Conics and the Calculus were recognized as admissible but not essential.

If for a moment we turn our attention to Saxony, we find that Analytical Geometry had been a recognized subject in the Gymnasium since 1875, and there was in the annual report of one Saxon school an actual reference to the notion of a function and its graphical representation, but in 1882 synthetic was substituted for analytical treatment in the official syllabus. It is, however, probable that the analytical methods were too deeply rooted to be at once eradicated by regulation.

At this period the great development in higher technical education was beginning to take effect in creating a demand in all schools for more graphical treatment, and the gradual introduction of this method may be clearly followed in the various editions of *Bardey's* text-books, the most popular mathematical school books in Germany. In 1881 appeared his Arithmetic and Algebra for middle schools, with an appendix dealing with graphs, to which in the preface he referred in the following words: "No pupil should leave a school of this type without knowing the meaning of graphical representation." In the eleventh edition of his *Graduated Exercises*, 1883, a section on graphical representations is added in the form of an Appendix, without any indication as to its use, and the words "dependent variable" are always used in place of "function."

An invitation by Hoffmann to discuss the question of graphical representation in his *Zeitschrift* met with no response, possibly owing to hesitation to

criticise the new Lehrplan so soon after its promulgation, or to the fact that public attention was riveted upon the struggle for recognition carried on by the Real Schools. It was, however, at this time becoming clearly evident that the idea of function was gradually being accepted as the ultimate goal of the teaching of Mathematics in Schools.

A new champion for the function-idea appeared in 1884, *Simon*, a master at a large composite 9-class school at Strasburg, who in a paper called "The Elements of Arithmetic and Algebra as introduction to Function Theory" insists that the true ultimate goal of these studies is the Theory of Functions and complex number, and only by constantly bearing this in mind can they be made to perform their true task—namely, the training of the logical faculties. Simon's view, as will be seen, is however by no means that of the reform party.

As an indication of the extent to which these new ideas had elsewhere penetrated into the official curriculum, I propose to give a free translation of the explanatory notes in the Austrian Programme of 1884 dealing with the introduction of Analytical geometry in U I (the top class but one).

"Obviously it is impossible to avoid introducing the ideas of constants and dependent and independent variables in the fundamental treatment of the subject; the branches of mathematics previously studied by the pupil offer ample opportunities for linking up these notions with those already familiar, and thereby making them intelligible. It will be a gain for Analytical Geometry if the pupil has become acquainted in the Trigonometry of the class below (O II) with Cartesian coordinates.

"The work should consist of the solutions of a series of problems which the pupil is called upon to master under the guidance of the teacher. A beginning should be made with representation of linear equations. The general problem of the graphical representation of a variety of simple forms of equations demanding no intricate investigations should precede a full treatment of the theory of straight lines. In this way the pupil can be made familiar with the discussion of equations and the geometrical consequences of their various forms. To save time and render the work more easy, the graphs should be drawn upon squared paper. For this purpose algebraical equations of the form $y^2=cx$, $x^2\pm y^2=c$, $ax^2\pm by^2=c$, $xy=a$, $y^m=ax^n$, $x^my^n=a$ should be employed, in which, of course, coefficients and indices are given particular values. As examples of transcendental curves $y=\log x$, $y=\sin x$ may be chosen; the mere sight of their graphs shows the course and variations of their functions more obviously and clearly than even the most thorough study of Tables.

"To emphasize the general application of this method graphs should be drawn illustrating some of the laws of Physics, while empirical functions—*e.g.* curves of mortality or temperature—should not be omitted. Quite apart from the value which these exercises have for the grasp of future teaching, the pupil becomes familiar with a method which may even be applied when seeking the general character of some not strictly mathematical law which holds good for a series of accurately observed results."

These remarks refer, however, to the introduction of graphical work in U I (the top class but one) and by no means to graphs as an item in the work of every class—the demand of the modern reformers.

It was in reference to these instructions that *Höfler*, formerly a schoolmaster, now Professor at the University of Vienna, and well known as the author of the standard German work on the teaching of Mathematics, wrote in 1887: "There is no better method of familiarising the pupil with the function-idea than by leading him to continual plotting of graphs of all kinds. I maintain that the habit of considering and mastering the subject-matter of elementary Algebra and the trigonometrical ratios expressly as examples of function should be the natural aim of the whole mathematical teaching in secondary schools." Höfler also points out the use of graphs

in dealing with logarithms as well as trigonometrical ratios, work properly assigned to the Oberstufe or upper stage. These, however, can always be led up to in the lower school in "the half-playful treatment of geometrical figures."

In the same year (1887) *Hoffmann*, in an editorial note in the *Zeitschrift*, called attention to the fact that there were practically no references to graphical work in any text-book at that time in use. "Henceforth, however," says he, "no new text-book and no new edition of an already existing text-book should appear without a treatment of the subject suitable for school use, that is to say, so arranged as to render abstract calculation easier for the pupil by means of geometry, which is much less abstract because more obvious." This, however, was only the first glimmering of the true reform gospel, and at that time such ideas met with by no means general acceptance.

In the year 1890 the question of the mathematics proper to a school course again came to the fore, and especially the point as to whether Conics should be included in the curriculum or not. At a meeting of teachers of Mathematics and Science, convened at the instigation of Hoffmann for the consideration of all kinds of reform proposals, the motion was brought forward that Conics, both analytical and geometrical, ought to form part of the work in the Gymnasium. But this suggestion was rejected, partly on the plea that the subject was more difficult than the elements of the Calculus.

In the same year at an official conference in Berlin, which included representatives of all parties interested in education, there was a general demand for the omission of certain parts of the mathematical course, and subsequently the Minister for Education invited certain experts to make suggestions for a new programme. Thus it came about that in the Prussian syllabus of 1892 a compromise was arrived at, and while transcendental equations, continued fractions and the treatment of Maxima were omitted, the opportunity was seized "to introduce the boys in the upper classes to the especially important idea of coordinates, and to explain to them in the simplest possible manner some of the fundamental properties of Conics." In this manner a way was opened for the introduction for the function-idea, although at that time no definite reference to it was made by any one. Saxony followed suit in the following year, and allowed a mathematical teacher to deal with coordinates when he deemed it expedient, at the same time definitely laying down a course of graphical representation of functions for the top class.

In criticising the Prussian syllabus, Simon, who had already championed the cause of the function-idea in a somewhat different manner, pointed out that if boys were to be taught "the especially important idea of coordinates," it was essential that they should *at an early stage* acquire a grasp of the function-idea—"the most far-reaching and indispensable idea of Algebra." Again, at a meeting in 1897 of the "Society for the Improvement of Teaching in Mathematics and Science"—a society generally referred to as the Förderungsverein and corresponding roughly to the English Mathematical Association, the proposal was made to introduce functions in U II (the fourth class from the top).

Finally, at a meeting of headmasters of the Rhine Provinces in 1899 the same demand was made for Gymnasium and Realschule, and it was unanimously resolved that the "ideas of coordinates and functions and the graphic representation of the course of a function should be introduced in the Realschule in U II and in the Gymnasium in I, and abundantly turned to account in the teaching of Mathematics and Science."

The consequence of these events became apparent when in the new syllabus of 1901, published after another official conference in Berlin, the following note appeared in reference to the revision work of the top class: "At this stage opportunity arises of ensuring a thorough comprehension of the idea of Function, with which the boys have already become acquainted earlier in the

course." Be it noted, however, that no definite mention of this notion was made in the prescribed subjects for each class, the inference being that mathematical teachers continually refer to the idea, not necessarily in specific terms, even in elementary work.

(*To be continued.*)

NOTICE.

PROFESSOR M. J. M. HILL, M.A., LL.D., Sc.D., F.R.S., will lecture at University College, London, on "The Theory of Proportion."

The principal object of the Lectures is to show that there exists a modification of Euclid's treatment of the Theory of Proportion, which does not require for its understanding anything more than a knowledge of elementary Algebra.

It is universally admitted that Euclid's treatment of the subject is far too difficult for inclusion in an elementary course of study.

The main reason for the difficulty of his argument is this:

The argument rests on two fundamental definitions, one of which is the test for determining when two ratios are equal, and the other is the test for distinguishing the greater from the smaller of two unequal ratios.

Euclid employs propositions depending on the second test in order to prove properties of equal ratios. This is unnecessary, and makes his proofs complex and indirect. The first test is by itself adequate to prove all the properties of equal ratios. The demonstrations of the propositions based on the first test, which will be given in these lectures, present a striking similarity to each other; but Euclid's demonstrations proceed in many cases on entirely different lines, it not being clear beforehand when it is necessary to use the first test, and when the second. Even those who can understand the steps of his argument taken separately find it difficult to obtain a comprehensive grasp of the subject as a whole.

The earlier Lectures of the Course will be devoted to an exposition of the subject in a simple and direct manner, with applications to Geometry, and the later Lectures to the consideration of Euclid's argument from a historical point of view, and its connection with modern ideas of number.

This course is open to teachers actually employed within the Administrative County of London, irrespective of the particular institution in which they may be engaged, on payment of a registration fee of 1/. Teachers who desire to avail themselves of the nomination of the London County Council must apply to the Education Officer, Education Offices, Victoria Embankment, W.C., on Form H 445 on or before January 1st, 1913. The Lectures will be given at University College, London, on Tuesdays at 6 p.m., commencing on Jan. 14th, 1913.

October, 1912.

CORRESPONDENCE.

Curepipe Road,
Ile Maurice.

MONSIEUR,

Confiant dans votre zèle pour la vérité et pour le progrès, je porte à votre connaissance les faits suivants:

Je puis prouver que la science actuelle des Champignons (Fungi) n'est qu'une vraie confusion où les transformations et les modifications d'un Champignon unique, souvent même chacune de ses parties constitutives, ainsi que les transformations de ses différentes sortes de spores, et, dans bien des cas, ces spores elles-mêmes, ont été classées comme espèces et réunies pêle-mêle en genres, familles, sous-ordres et ordres, ou de toute autre manière selon la fantaisie des auteurs. C'est pour cette raison que la Mycologie actuelle est si difficile, si obscure, et si nulle en résultats pratiques. C'est pour cette raison que la Science est incapable de trouver la solution du problème des Origines.

Je puis prouver que chaque modification et chaque combinaison nouvelles de substances (certaines des inorganiques, en très minimes quantités) produisent de nouvelles espèces de champignons.

Cette grande vérité: "Il n'y a qu'Un Champignon, qui se transforme à l'infini selon le milieu," constitue la découverte du Protéisme ou Unité du Champignon (Identity of all true Fungi).

Principaux avantages de cette découverte:

L'Unité du Champignon établit définitivement la philosophie de l'Univers;

En donnant la solution du problème des Origines, elle prouve d'une manière absolue l'existence d'un Dieu Créateur, et celle d'un principe de vie distinct de la matière, simple, et par conséquent immortel;

Elle prouve le pouvoir de faire naître a volonté de nouvelles espèces animales et végétales;

Elle met à notre disposition une nouvelle analyse, qui sera pour les substances organiques ce que l'analyse spectrale est pour les inorganiques;

Elle permettra de savoir de quelles formes du Champignon naissent les microbes pathogènes, et nous donnera les moyens de les supprimer ou de les mieux combattre, etc.

Comme vous pouvez en juger par cet aperçu, c'est là une découverte dont on ne peut se passer; mais, comme elle va détruire des systèmes fameux et une foule d'erreurs, elle rencontre cette hostilité des corps savants qui n'a que trop souvent retardé les découvertes les plus utiles, lorsqu'elles étaient contraires aux idées reçues.

L'Unité du Champignon étant évidente a priori, et *pouvant être très facilement vérifiée par l'expérience, de mille manières*, n'a pas à craindre une opposition ouverte, ni une discussion quelconque: elle n'a besoin que d'être connue.

Je suis pauvre, comme beaucoup de chercheurs l'ont été avant le succès, et je n'ai pas les moyens de payer une campagne de publicité; mais je prends dès maintenant l'engagement d'abandonner toute récompense pécuniaire, jusqu'à concurrence de Quatre mille livres sterling, à celui ou à ceux qui selon moi auront le plus contribué à la faire admettre.

Je vous en enverrai volontiers la démonstration, si vous vous voulez bien me promettre de renseigner le public et de vulgariser l'Unité du Champignon *aussitôt votre conviction dûment établie.*

Je vous demande ce service au nom de la vérité, et au nom du progrès humain.

Agréez, Monsieur, l'assurance de ma parfaite considération.

LACHICHE HUGUES.

Cette découverte doit vous intéresser tout particulièrement, parce qu'une des conséquences de ce grand "fait nouveau" est de détruire ce que l'on appelle "La Cosmogonie de Laplace" en prouvant que "l'inorganique vient de l'organique," et que "c'est par la Vie que Dieu a formé le Monde." *Cela s'accorde parfaitement avec votre découverte intéressante des "Somersaulting Planets."*

A Mr. le Professeur H. H. TURNER,
President of the Mathematical Association, London.

DIRGE.

Sung to Lamentations at the Dinner of the Mathematical Association, Jan. 1912.

(AIR—*Tom Bowling.*)

Here, a mere book, lies poor Euclides,
The darling of our schools;
We do not doubt his *bona-fides*,
But now we've found new tools.
His works were read like *Aesop's Fables*
Full many a time and oft;
Ages they graced all schoolroom tables,
But now they're in the loft—
They now adorn the loft.

When Euclid framed his definitions
He did not miss "the point";
Space was prescribed by his conditions
For angles twain conjoint.
Our fathers, aye! and theirs before 'em,
At Euclid seldom scoffed,
They feared the dread Pons Asinorum,
Which now is in the loft—
It now adorns the loft.

But still, in spite of Time's abuses,
This central truth abides,
That squares of all hypotenuses
Are sums of squares on sides.
We verify such facts by measure!
Though Euclid's proofs be doffed,
The gems he set we safely treasure,
The setting's in the loft—
It now adorns the loft. H. H. T.

REVIEWS.

Le Calcul des Probabilités et ses Applications. By E. CARVALHO. Pp. ix+169. 6 frcs. 50. 1912. (Gauthier-Villars.)

As an examiner for the French Statistical Service, M. Carvalho has found that the student who has received no more than an elementary knowledge of mathematics rarely, if ever, is able to get from the usual sources a real grip of the theory of probabilities. The volume before us is intended for people of general culture who have never specialised in mathematics. The fundamental principles are clearly and simply expounded, and applications are illustrated without the usual parade of analytical developments required in the establishment of formulae. The author reminds us that Hermite used to say that just as there are natural phenomena, there are such things as mathematical facts. It is quite unnecessary for the average politician or statistician to know more than the fact. Now the theory of Probabilities, as ordinarily expounded, consists "of little else than artificial transformations and complicated formulae encumbering the mind without revealing the reason of things." There is, for example, no earthly reason why ordinary folk should be able to prove Stirling's Theorem. It is certainly necessary to the theory, but the practical application of it is all that is really required. We can well imagine that this volume would be an excellent introduction to the subject for the young mathematician who is afterwards to pursue the study of probabilities along the normal lines. Particular mention should be made of the carefully selected statistical examples discussed in the text. The book is brought to a close by a useful discussion on the limits of the theory, and the precautions that must be taken in its application to questions which are incomplete or ambiguous.

Paul Tannery. Mémoires Scientifiques. Edited by J. L. HEIBERG and H. G. ZEUTHEN. I. *Sciences Exactes dans l'Antiquité.* 1876-1884. With Portrait. Pp. xiii+465. 15 frcs. 1912. (Gauthier-Villars.)

In his early days Paul Tannery was deeply interested in the pedagogic side of Mathematics, but he published nothing on the subject. We are reminded of the fact by his brother Jules in the Preface to *Notions de Mathématiques*, a volume to which, it will be remembered, the elder brother added a few pages of *Notions Historiques*. M. Painlevé has felicitously remarked that if Littré was a lay saint, Jules Tannery was an intellectual saint. His deep and subtle intellect found itself at home in criticism both scientific and philosophical. We may be sure that he weighed well his words when he wrote of his brother Paul in the Preface we have mentioned: "Sans parler de tout ce qui m'attache à lui, je ne pouvais trouver chez personne une science plus sure, un esprit plus large et plus philosophique."

It was about the middle of the last century that Paul Tannery fell under an influence which was almost incidentally to determine the direction his life's work was to take. He has left it on record that it was in order to follow up the train of thought suggested by something written by Comte that he began the study of the history of Science. He was never in the ordinary sense of the word a Comtist, though he looked upon himself as a spiritual disciple of that inspiring master. Perhaps the influence of Lachelier during his school days gave him a taste for Philosophy, but in any case the taste was there, and flourished in spite of its incongruity with his early occupations. After leaving the École Polytechnique—which he entered at the age of 17—he was appointed to a post in the École d'Application des Tabacs. He steadily rose in this Department, and eventually attained high office. Above the Table of Contents of the volume before us is a vignette—crossed swords and the interwreathing motto *en place et lieu*. This reminds us that Paul Tannery was in Paris at the time of the siege, being Captain in command of a battery of mitrailleuses. He attained the highest rank he reached—Lieutenant-Colonel in the Artillery of the Territorial Army—ten years before his death. After the Commune he resumed his work in connection with the State tobacco monopoly, but was overtaken by a serious illness followed by a long period of convalescence. This he occupied by acquiring a mastery of Latin and Greek. He was already no mean linguist, *e.g.* his knowledge of Hebrew was sufficient to enable him to make a translation of a considerable portion of the Old Testament. It is interesting to note with respect to his official labours, and his relations with the working men who came under his rule, that he was universally beloved and respected for his straightforward nature, his kindness of heart, and the manner in which his tact and judgment enabled him to find a compromise amid conflicting interests. The illness to which we have alluded marked a turning-point in his career. While he continued to devote himself to his work as a civil servant, he now gave up his evenings from 8 p.m. to 1 a.m. to research, mainly in the history of Science and Philosophy in Antiquity, the Middle Ages, and the sixteenth and seventeenth centuries. An anonymous writer in the *Revue Générale des Sciences* has dilated upon the consummate knowledge he gradually acquired of those periods, to his untiring patience, his scrupulous and conscientious attention to detail, to his memory, which was marvellous in its extent and tenacity, to his habit of verifying every reference himself, and to the independence of judgment which left him indifferent to current views and generally received opinions. He was the first Frenchman to apply to the study of Ancient Philosophy a method both historic and concrete. Hitherto the labours of Zeller had served as a model for work of the kind, best described, perhaps, as a far-reaching system of interpretative construction. But a new school was arising with different ideals. The revolt against the Zellerian tradition was led by a few Germans, and Teichmüller in particular, who endeavoured to show how philosophical and scientific systems, and even the most fragmentary portions of the thought and belief of an age sprang from its social conditions, its traditions, and its customs. To any set of doctrines they attributed not merely the value they possess to us and to the history of thought, but the actual value they possessed for the men who created them and believed in them. Thus, piercing the veil of legend, they drew a vivid picture of real living men, responsive to the moral and physical demands of their time, and working for the moral and intellectual needs of their contemporaries. Paul Tannery contributed independently an impulse to such researches. His first papers on the mathematical teachings of Plato appeared before he had seen or heard of the work of Teichmüller—work which he made known to his compatriots, and which by his own discoveries he was able to supplement and enlarge.

Historical research was but part of his intellectual activity. In the days when he was more particularly interested in teaching, he had already begun to consider the philosophy of mathematics. At a time when few men in France were qualified on both sides to undertake such investigations, he was writing articles in the *Revue Philosophique*, in which we see his endeavour to exhibit psychological reality, the living operations of which mathematical notions are the residue, and his anxiety to dissipate the existing vagueness of views by giving explanations that explain. Among the first articles he published were those on the Nuptial Number in Plato, the astronomical system of Eudoxus, the date of Diophantus. These are the first of some 400 Notes, Memoirs, etc., contributed to a couple of dozen periodicals

In addition to these he wrote volumes on Greek Science, from Thales to Empedocles; on Greek Geometry; on Ancient Astronomy; on the unpublished letters of Descartes; and a text and Latin translation of Diophantus. With Henry he edited vols. i., ii., iii. of the collected edition of the works of Fermat, and with Adam, the first seven and the ninth volumes of the great edition of Descartes. He contributed chapters on the History of Science to the *Histoire Générale*, published by MM. Lavisse and Rambaud, and a large number of historical notes to the French edition of the *Encyclopédie mathématique*. The volumes of *Mémoires* will contain nothing from any of his published volumes, nor his contributions to the *Intermédiaire des Mathématiciens* and the *Bibliotheca Mathematica*. A selection will be included from his critical and biographical articles in the *Grande Encyclopédie*. The remaining volumes of the *Mémoires* will appear under the following headings: Exact Sciences among the Byzantines, in the Middle Ages, and in modern times; Pure Mathematics; Philosophy; Classical Philology; and a volume of "Recensions." Later on it is proposed to issue a volume with a biography, a section devoted to bibliography, and another containing a selection from his scientific correspondence. The titles of a few of the articles in the present volume will indicate the nature of the pabulum it offers to the reader: Pappus and Hero on the Arithmetic of the Greeks; Archimedes' indeterminate problem of the oxen of the Sun;* Fragments from Apollonius of Perga; The Fragments of Hero preserved by Proclus; The Geometrical Solutions of Problems of the Second Degree before Euclid; Aristarchus of Samos; Hero's Stereometry; On the Modius Castrensis.

It has been said that the merits of Paul Tannery were better recognised abroad than at home. That two famous Danish savants should have undertaken the preparation of this handsome tribute to his memory is in itself no ordinary compliment. It is sad to remember the great disappointment that befell Paul Tannery towards the end of his life. For five years he had been Professor of Greek and Latin Philosophy. When the Chair of the History of Science fell vacant, Tannery's name was proposed to the Minister of Public Instruction by the practically unanimous vote of the Professors of the *Collège de France* and the members of the *Académie des Sciences*. Whatever the cause, the choice of the Minister fell upon another, of whom it was bitingly said, that, quâ historian of science, he is as unknown to-day as he was then.

It may be appropriate to close this notice by indicating as briefly as may be Tannery's judgment as to the principles to be kept in view in the teaching the history of science. They are set forth in a programme drawn up at the request of the Director of Secondary Instruction in France some twenty years ago, at the time when the organisation of the schools was the question of the day. He held that the main object of the teacher should be to set forth as clearly as possible the links which connect the evolution of each science with other sciences and with civilisation in general. The programme is divided into periods. The order of ideas, true or false, dominating each science in each of these periods, must be clearly shown and thoroughly grasped, as well as the nature of the transformations such an order of ideas underwent during the course of the period. It is needless to insist on a chronological treatment. It is better to keep to the general characteristics of each period. To this there is, of course, the exception, that a teacher may wish to refer back to the germs of great ideas and new discoveries, and at the same time to indicate the results of those discoveries in cases where he has no intention of returning to them in dealing with another period. While trying, as far as possible, to instil into the minds of students general ideas, it is well to maintain their attention by selecting from a number of subjects given in the programme one for each lesson. Circumstantial details may be grouped around this in such a way as to illustrate the general idea. The teacher is at liberty to choose the subject according to his individual taste, the sole proviso being that he must connect it clearly with the order of general ideas, which it is the object of the lesson to bring home to the class. In all cases meaningless and question-begging nomenclature must be avoided. It should be remembered that historical exposition, if too summary, fails in the precision it aims at, and often leaves quite incorrect notions in the minds of the students. If he prefers, the teacher may select the life of some great man instead of the subject—Watt, for example,

* Cf. Sir T. L. Heath, *Diophantus of Alexandria*, pp. 121-4.

instead of the steam-engine. In that case he must not omit to give such an analysis of his most important works as will arouse the interest of the pupils sufficiently to induce them to go to the original sources for further information. He must never lose sight of the fact that the historical study of science is as much concerned with ideas that are false as with those that are true. The real importance of science is grasped only by those who have a wholesome appreciation of its errors in the past. It is needless to say that full opportunity must be taken of the natural interest offered by the practical applications of science. But no opportunity must be lost of enforcing the view that science alone can lead to accurate conceptions either of the universe or of human society. It will be seen that if Tannery's scheme were carried out in its entirety, the question of the day would have rapidly resolved itself into that of the training of teachers. Teachers of mathematics in this country may be attracted by some of its features. As for errors, it may be worth while reminding them that during the last year or so notes on the mistakes made by great mathematicians have been appearing in the *Intermédiaire des Mathématiciens*.

THE LIBRARY.

THE Librarian acknowledges with thanks the receipt of two more books, presented to the Library by Mr. W. Gallatly. Also the gift, by the author, of Muir's *History of Determinants*, in two volumes.

The Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2	copies of *Gazette*	No. 2 (very important).
2 or 3	copies of Annual Report	No. 11 (very important).
1 or 2	„ „	Nos. 10, 12 (very important).
1 copy	„	Nos. 1, 2.

ERRATUM.

P. 382, l. 28, for "Rowth" *read* "Routh."

BOOKS, ETC., RECEIVED.

Analyse Vectorielle Générale. I. *Transformations.* By C. BURALI-FORTI and R. MARCOLONGO. Translated into French by P. BARIDON. Pp. xix + 179. 6 frcs. 1912. (Hermann, Paris; Bowes & Bowes.)

Exercises in Modern Arithmetic. By H. SYDNEY JONES. Pp. 336. 2s. 6d. 1912. (Macmillan.)

Differential und Integralrechnung. Vol. I. *Differentialrechnung.* By W. F. MEYER. 2nd edition, revised and enlarged. Pp. xv + 418. 9 m. 1912. Sammlung Schubert, x. (Göschen.)

Nichteuklidische Geometrie. By H. LIEBMANN. 2nd edition, revised. Pp. vi + 222. 6.50 m. 1912. Sammlung Schubert, xlix. (Göschen.)

Darstellende Geometrie. By T. SCHMID. Vol. I. Pp. vi + 279. 7 m. Sammlung Schubert, lxv. 1912. (Göschen.)

The Pell Equation. By E. E. WHITFORD. Pp. iv + 193. $1, post free. (College of the City of New York.) 1912. A history of the equation $x^2 - Ay^2 = 1$ table of solutions from $A = 1501$ to $A = 1700$. Bibliography with reference to over 300 authors. Table of continued fractions for $\sqrt{A}$.

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